Classification of invariant Fatou components.

Theorem: Let $f: \hat{\mathbb{C}} \circlearrowleft$ be a rational map, and $U = f(U)$ an invariant Fatou component. Then one of the following cases occur:

(1) U is the immediate basin of attraction of a contracting fixed point.

(2) U is the immediate basin of attraction of a parabolic fixed point with multiplier $\lambda = 1$ (associated to an attracting direction / petal).

(3) U is a Siegel disk

(4) U is a Herman ring.

Proof:

By previous results, we already know that we must have one of the following cases:

(a) U contains ~~an~~ a contracting fixed point $\Rightarrow$ (1)

(b) all orbits of U converge to a boundary fixed point.

(c) $f|_U$ is an automorphism of finite order : cannot occur for $d \geq 2$ ($\Rightarrow f^m = id$)

(d) $f|_U$ is conjugate to an irrational rotation on a disk, a punctured disk or an annulus: (d1) $\Leftrightarrow$ Siegel (3), (d3) $\Leftrightarrow$ Herman (4), (d2) cannot occur because $J(f)$ has no isolated points.

Hence we only have to prove that, in case (b), ~~the set on~~ the boundary fixed point p must have multiplier $\lambda = 1$.

Clearly p is not contracting, nor a Siegel point ($p \in J(f)$), nor repelling, since it attracts points in U. Hence $|\lambda| = 1$

Up to change coordinates, assume $p=0$, $f(z)=\lambda z(1+o(1))$.

Pick any point $z_0 \in U$, and a continuous path $\gamma:[0,1]\to U$ between $\gamma(0)=z_0$ and $\gamma(1)=f(z_0)$. Extend γ to $\gamma:[0,+\infty)\to U$ by setting recursively (on $[t]$) $\gamma(t+1)=f(\gamma(t))$.

(We say that γ "converges to z_0") (Snail lemma).

Suppose by contradiction that $|\lambda|=1$ but $\lambda\neq 1$. ~~We could also~~

When $t\to\infty$, $\gamma(t)\to z_0$, and the linear part of f dominates the action.

In particular we have the asymptotic behavior $\gamma(t+1)\sim\lambda\gamma(t)$ as $t\to+\infty$.

If γ does not selfintersect, it looks like a spiral

Taking a radial segment between two adjacent turns of the spiral gives a region W which is mapped strictly into itself: $f: W\to W$, $f(W)\subsetneq W$.

By Schwarz lemma, 0 must be attracting, against the hypothesis $|\lambda|=1$.

In the general situation, consider polar coordinates (ρ,θ) in $\mathbb{C}^*$, and lift $\gamma(t)$ to $\tilde\gamma(t)=(\rho(t),\tilde\theta(t))$ in the universal covering $\mathbb{R}_+^*\times\mathbb{R}$ of $\mathbb{R}_+^*\times\mathbb{R}/_{2\pi\mathbb{Z}}$.

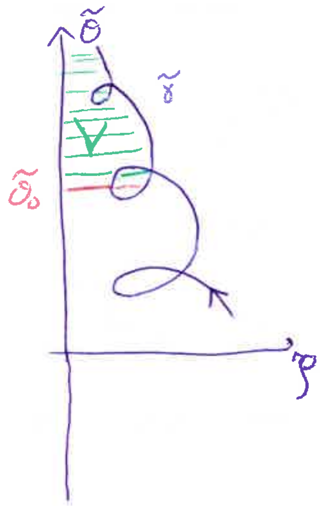
Since γ converges to 0, $\rho(t)\to 0$ when $t\to\infty$, and $\tilde\theta(t+1)=\tilde\theta(t)+c+o(1)$, where $c\in\mathbb{R}$ is a constant satisfying $e^{ic}=\lambda$. (from $\gamma(t+1)\sim\lambda\gamma(t)$)

Similarly, for $r_0 \ll 1$, f is univalent on D_{r_0}, and lifts to a map $\tilde f(r,\alpha)=(\tilde r(r,\alpha),\tilde\alpha(r,\alpha))$, defined and univalent for $r<r_0$, $\alpha\in\mathbb{R}$.

Notice that $\tilde r=r(1+o(1))$ and $\tilde\alpha=\alpha+c+o(1)$, with c the same as before

as far as we pick the right lift $\tilde{f}$ of f.
(This comes from the fact that $f(z) = \lambda z(1+o(1))$.

We ~~...~~ want to prove that $c = 0$.
Assume that $c > 0$ (the case $c < 0$ is completely analogous).
Chose $r_1 \ll r_0$ so that $\tilde{\alpha} > \alpha + \frac{c}{2}$ $\forall r \leq r_1$.
Then pick t_0 so that $\rho(t) \leq r_1 \Rightarrow \tilde{\theta}(t+1) > \tilde{\theta}(t) + \frac{c}{2}$ when $t > t_0$
Finally, pick $\tilde{\theta}_0$ so that $\tilde{\theta}(t) < \tilde{\theta}_0$ for $t \leq t_0$.
Consider now the connected region V in the plane $(\rho, \tilde{\theta})$
~~the~~ bounded by $\tilde{\theta} = \tilde{\theta}_0$, $\rho = 0$ and $\tilde{\gamma}([t_0, \infty))$
$\tilde{f}$ maps V univalently into itself, and the
ρ coordinate tends to zero under iteration.
This implies that $|f^n(z)| \to 0$ in a neighborhood of 0, and 0 is attracting,
against the assumption $|\lambda| = 1$.



$\square$

This theorem describes quite precisely the dynamics of rational maps on fixed, or more generally periodic and preperiodic Fatou components.
Sullivan's ~~st~~ no wandering theorem ~~...~~ states that all Fatou components of a rational map are preperiodic, and allows us to fully understand the dynamics on the Fatou set.
While the existence of ~~pa~~ attracting or parabolic basins is quite easy to achieve, the existence of Siegel disks comes from

the local study of irrational germs and diophantine conditions, while the existence of Herman rings requires more work. Both results will be proven in the next lessons, by techniques of quasi-conformal deformation/surgery.

Rem: The situation for transcendent maps in $\mathbb{C}$ is quite more involved. In fact, there exist maps with wandering domains (e.g. $f(z) = z + \sin 2\pi z$), and there are other types of invariant Fatou components. In fact, in case (b) of the list in the previous theorem, we could have the boundary point to be the essential singularity at ∞, and the dynamics in this case looks quite different. Such case is called a Baker domain.